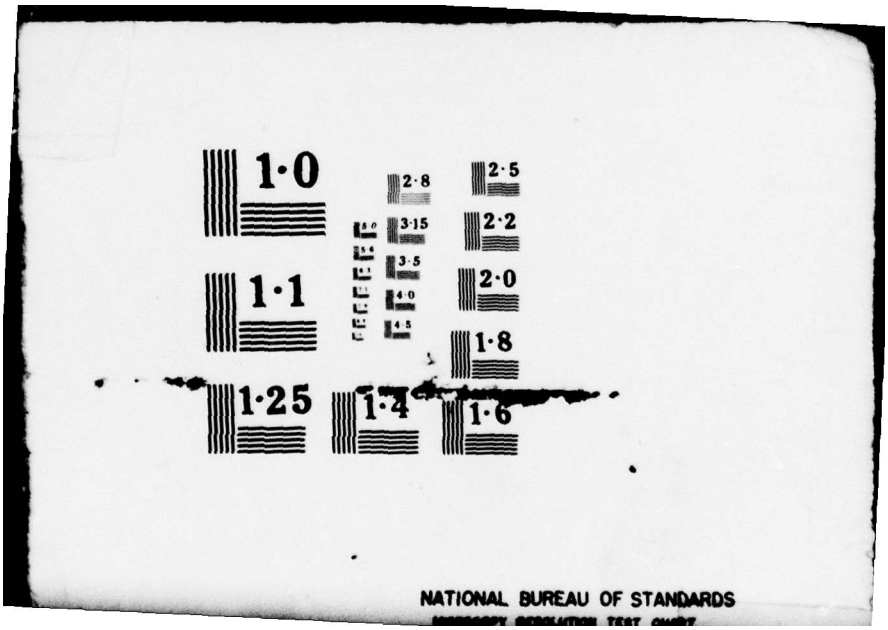
AD-A072 921 AIR FORCE INST OF TECH WRIGHT-PATTERSON AFB OH SCHOO--ETC F/G 12/1
SPECTRAL ANALYSIS OF TIME SERIES.(U)
MAR 79 A L BERTAPELLE
UNCLASSIFIED AFIT/GST/MA/79M-1 NL

1 OF 1
AD
A072921

END
DATE
FILMED
9-79
DDC


1·0
1·1
1·25
1·4
1·6
1·8
2·0
2·2
2·5
2·8
3·15
3·5
4·0
4·5
NATIONAL BUREAU OF STANDARDS

LEVEL II

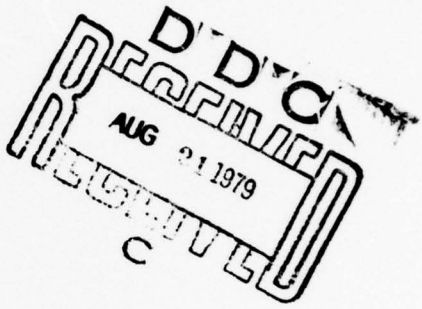
UNITED STATES AIR FORCE

AIR UNIVERSITY

AIR FORCE INSTITUTE OF TECHNOLOGY

Wright-Patterson Air Force Base, Ohio

AFIT/GST/MA/79M-1



SPECTRAL ANALYSIS OF TIME SERIES

THESIS

AFIT/GST/MA/79M-1 Anthony L. Bertapelle
Captain USAF

Approved for public release; distribution unlimited

(14) AFIT/GST/MA/79M-1

(6) SPECTRAL ANALYSIS OF TIME SERIES,

(9) Master's thesis,

THESIS

Presented to the Faculty of the School of Engineering
of the Air Force Institute of Technology
Air University
in Partial Fulfillment of the
Requirements for the Degree of
Master of Science

(12) 59 p.

(10) by
Anthony L. Bertapelle, ~~MBA~~
Captain USAF

Graduate Strategic and Tactical Sciences

(11) March ~~9th~~ 1979

012 225

Preface

The motivation behind this thesis is a query from AFTEC and an interest expressed by Major Richard W. Kulp. I could see applications in modeling so I thought it would be used by others. One of my criteria for a thesis was a subject that could be useful to the United States Air Force.

I owe Major Kulp a lot for guiding me through the basics of Time Series Analysis since I had not had any formal instruction in it. His encouragement and help kept me on track. Without him this thesis would still be waiting to be written. Thanks also go to David R. Brillinger, even though he does not know me; his book is invaluable. I have read at least half of it and skimmed the rest at least a dozen times. Also to P. R. Krishnaiah who encouraged me when he was at Wright-Patterson Air Force Base to give a talk on multiple time series.

I cannot forget my wife, Connie, who kept the children from my books and papers, fed the family on TV trays when I took over the dining room table, helped me count failures and passes in the simulation runs, and typed draft copies until she knew what I was saying.

Do not blame any of the above for any errors or omissions. I take full responsibility for this paper.

Anthony L. Bertapelle

Contents

List of Figures

List of Tables

Abstract

One use of spectral analysis of time series is to determine if two different time series are realizations from the same process. This thesis develops the theory behind Krishnaiah and Schuurmann's theoretical work reported in their report Approximations to the Distributions of the Traces of Complex Multivariate Beta and F Matrices. We take the trace of a test statistic calculated from the spectral density matrices of the time series and test it. The thesis applies the theory to two small sample simulations.

SPECTRAL ANALYSIS OF TIME SERIES

I. Introduction

The Air Force Testing and Evaluation Center (AFTEC) at Kirtland Air Force Base, New Mexico, has had a requirement in the past to compare two different time series to determine if it can be assumed that they are realizations from the same process. They also foresee having the same requirement in future tests where they will be comparing radar tracking errors. One time series will come from a simulation model, the Air Force Electronic Warfare Environmental Simulation (AF-EWES) model developed by General Dynamics, Fort Worth, Texas. The second time series will be determined from field tests.

Each of the error series is a three variable, vector time series. Each variable represents the error in a specific direction of a three-dimensional reference surface. That is, one variable represents the error in the x-direction, another variable represents the error in the y-direction, and the third variable represents the error in the z-direction.

This technique will be useful in validating the results of a simulation model with real-world data. While one can not prove that a model is correct, one can collect evidence

(data) that may support the model or show that the model does not fit the data.

This thesis will describe one use of spectral analysis to test two hypotheses. The procedure used was developed by P. R. Kirishnaiah and F. J. Shuurmann. The null hypothesis is that the two time series being compared can be considered realizations from the same process.

The organization of the thesis is as follows. Chapter II reviews the notation, theory and procedure used in comparing the time series. Chapter III contains the results of Monte Carlo simulation while conclusions and recommendations are in Chapter IV.

II. Theory and Method

Terminology and Notation

In the following we will assume that $\{\underline{X}(t), t=0, \pm1, \ldots\}$ is an r-variate Gaussian time series with $E(\underline{X}(t)) = \underline{0}$ for all t. If $E(\underline{X}(t)) = \underline{\mu}$ for all t then we always write $\underline{Y}(t) = \underline{X}(t) - \underline{\mu}$, $t=0, \pm1, \ldots$ and the theory will hold for $\underline{Y}(t)$. $\underline{X}(t)$ is said to be covariance stationary if for all integers s and t,

(i). $E(\underline{X}(t)) = \underline{\mu}$, all $t \in I$, the set of all integers

(ii). $E(\underline{X}(t)\ \underline{X}'(s)) = \underline{\Gamma}(t,s) = \underline{\Gamma}(0,s-t) = \underline{\Gamma}(h)$

Condition (i) says that the mean vector is constant over the index set and condition (ii) says that the covariance between observation $\underline{X}(s)$ and $\underline{X}(t)$ depends only on the difference between s and t and not on the values of s and t.

When we write $\underline{X}(t)$ we mean

$$\underline{X}(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \\ \vdots \\ X_r(t) \end{bmatrix} \qquad (1)$$

where the $X_r(t)$ are scalar time series. Bars under capital letters generally indicate matrices. If a matrix $\underline{A}$ has

elements a_{jk} we may indicate the matirx as $[a_{jk}]$, i.e., $\underline{A} = [a_{jk}]$, where j indicates the row position of a and k the column. The expected value of $\underline{X}(t)$ is denoted

$$E(\underline{X}(t)) = \begin{bmatrix} E(X_1(t)) \\ E(X_2(t)) \\ \vdots \\ E(X_r(t)) \end{bmatrix} = \underline{\mu} \qquad (2)$$

We will estimate the expected value of $\underline{X}(t)$ using the arithmetic mean,

$$\bar{\underline{X}} = \frac{1}{T} \sum_{t=1}^{T} \underline{X}(t), \qquad (3)$$

where T is the number of data points in each scalar time series.

Denoting the transpose of $\underline{X}(t)$ as $\underline{X}'(t)$ we define the covariance matrix of $\underline{X}(t)$ and $\underline{X}(t + h)$ as

$$\underline{\Gamma}(h) = E(\underline{X}(t)\ \underline{X}'(t + h)) = [\gamma_{ij}(h)] \qquad (4)$$

where $\gamma_{ij}(h) = E(X_i(t)\ X_j(t + h))$. $\underline{\Gamma}(h)$ must be absolutely summable, that is,

$$\sum_{h=-\infty}^{\infty} |\gamma_{ij}(h)| < \infty \quad \text{for } i, j=1,\ldots,r \qquad (5)$$

The sample covariance matrix will be denoted as $\underline{C}(h)$ and defined to be

$$\underline{C}(h) = \frac{1}{T} \sum_{t=1}^{T-h} E((\underline{X}(t) - \bar{\underline{X}}(t)) (\underline{X}'(t + h) - \bar{\underline{X}}(t))) \qquad (6)$$

where T is the number of data points for each vector and $\bar{\underline{X}}$ is the estimated mean of $\underline{X}(t)$, Eq (3). Fuller states this estimator has the smaller mean error of two biased estimators he proposed (Ref 4:236). He recommends using this estimator because it will ensure that the estimated covariance function is positive definite.

To talk about the correlation matrix for a vector time series we need to define it for a scalar time series. Let $\rho(h)$ be the autocorrelation of a stationary time series, then

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} \qquad (7)$$

To extend to a vector time series we have

$$\underline{P}(h) = \underline{D}_0^{-1} \sqcap(h) \underline{D}_0^{-1} \qquad (8)$$

where $\underline{D}_0$ is a diagonal matrix with the square root of the variances of $X_r(t)$ as the diagonal elements. $\underline{D}_0^2 = \{\text{diag}$ $(\gamma_{11}(0), \gamma_{22}(0), \ldots, \gamma_{rr}(0))$. We will write the ij^{th} element of $\underline{P}(h)$ as $\rho_{ij}(h)$ and call it the cross correlation between $X_i(t)$ and $X_j(t)$.

Theory

The theory used in this thesis will draw on terminology and notations defined in the previous section. As other terms are needed they will be defined. Proofs will not be given; the reader may find the proof or a discussion of the proof in the references. The procedure we will be using involves the calculation of a Fourier transform on the covariance matrix. Fuller uses two theorems to state a special case of the Fourier integral theorem (Ref 4:107-109 and 110-111). The theorems follow another theorem he presents that we need (Ref 4:106).

Theorem 1:

Let $f(x)$ be an absolutely integrable function of period 2π. Then,

(i) At a point of continuity where $f(x)$ has a right derivative and a left derivative,

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos k_x + b_k \sin k_x);$$

(ii) At every point of discontinuity where $f(x)$ has a right and a left derivative,

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos k_x + b_k \sin k_x) = \frac{f(x^+) + f(x^-)}{2}$$

In this theorem the Fourier coefficients, a_k and b_k are defined as follows (Ref 4:95-96).

Let L[A] = largest integer $\leq \frac{A}{2}$ then for $\omega_m = \frac{2\pi m}{N}$

$m = 0, 1, 2, \ldots, L[N]$ we have

$$a_m = \frac{2 \sum_{t=0}^{N-1} f(t) \cos \omega_m t}{N} \qquad m = 1,2,\ldots, L[N-1]$$

$$= \frac{\sum_{t=0}^{N-1} f(t) \cos \omega_m t}{N} \qquad m = 0 \text{ and}$$

$$m = \frac{N}{2} \text{ if } N \text{ is even}$$

and

$$b_m = \frac{2 \sum_{t=0}^{N-1} f(t) \sin \omega_m t}{N} \qquad \text{for } m = 1,2,\ldots, L[N-1]$$

Theorem 2:

Let f(x) be a continuous periodic function of period 2π with derivative f'(x) that is square integrable. Then the Fourier series of f(x) coverges to f(x) absolutely and uniformly.

Theorem 3:

Let the correlation function $\rho(h)$ of a stationary time series be absolutely summable. Then there exists a continuous function $f(\omega)$ such that:

(i) $\rho(h) = \int_{-\pi}^{\pi} f(\omega) \cos \omega h \, d\omega$

(ii) $f(\omega) \geq 0$

(iii) $\int_{-\pi}^{\pi} f(\omega)\, d\omega = 1$

(iv) $f(\omega)$ is an even function.

Theorem 2 states that a sequence of Fourier coefficients for a continuous periodic function with a derivative that is square integrable can be used to build a sequence of functions that coverges to the original function. The argument that follows is from Fuller (Ref 4:115-116). The result of Theorem 2 can be restated compactly by substituting the definitions of a_k and b_k into the statement of Theorem 1 and getting

$$f(x) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{-ikx} \int_{-\pi}^{\pi} f(\omega)\, e^{ik\omega}\, d\omega \qquad (9)$$

The reciprocal relationships are well defined and we can write Eq (9) in terms of Theorem 3.

$$\rho(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik\omega} \sum_{h=-\infty}^{\infty} \rho(h)\, e^{-ih\omega}\, d\omega \qquad (10)$$

We call $\rho(k)$ and $f(x)$ a transform pair.

We will call the transform with the constant and the negative exponential the Fourier transform or spectral density. The $f(\omega)$ mentioned in Theorem 3 defined by

$$f(\omega) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \rho(h)\, e^{-i\omega h} \qquad (11)$$

is the Fourier transform of $\rho(h)$. We can also say that it is the spectral density associated with $\rho(h)$. The second transform in Eq (10) consists of the constant and the positive exponential and is called the inverse transform or characteristic function. We can write

$$\rho(h) = \int_{-\pi}^{\pi} f(\omega)\, e^{ih\omega}\, d\omega \qquad (12)$$

We have seen that if we have a correlation function for a time series we can get a spectral density, and that if we have a spectral density we can find a correlation function that is the inverse transform of the spectral density. We can extend this discussion to the covariance function since the covariance function is the correlation function times the variance of the process. We will, again, follow Fuller's discussion (Ref 4:126-127). First, $\rho(h)$ can be expressed in a Lebesgue-Stieljes integral form as

$$\rho(h) = \int_{-\pi}^{\pi} e^{i\omega h}\, dG(\omega) \qquad (13)$$

where $G(\omega)$ is a statistical distribution function. If we multiply $\rho(h)$ by the variance of the process we can get the covariance function

$$\gamma(h) = \int_{-\pi}^{\pi} e^{i\omega h}\, dF(\omega) \qquad (14)$$

where $dF(\omega) = \gamma(0)\, dG(\omega)$. Both $G(\omega)$ and $F(\omega)$ have been called the spectral distribution function.

Now, assume $\gamma(h)$ is absolutely summable, as in Eq (5). By Theorem 3 we can define $f(\omega)$ by

$$f(\omega) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \gamma(h)\, e^{-i\omega h}$$

$$= \frac{1}{2} \sum_{h=-\infty}^{\infty} \gamma(h) \cos\omega h \qquad (15)$$

which is a continuous nonnegative even function and the inverse transformation is

$$\gamma(h) = \int_{-\pi}^{\pi} f(\omega)\, e^{i\omega h}\, d\omega \qquad (16)$$

and when we compare Eqs (16) and (14) we can say $dF(\omega) = f(\omega)\, d\omega$ and $f(\omega)$ is the spectral density function defined in Eq (11).

The discussion so far has been in terms of scalar time series. The spectral representations of vector time series follows in a straightforward manner. Again our discussion is from Fuller (Ref 4:153-154).

Recall that Eq (4) stated that

$$\Gamma(h) = [\gamma_{jm}(h)]$$

and assume that $[\gamma_{jm}(h)]$ is absolutely summable as defined

in Eq (5). Then,

$$f_{jm}(\omega) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \gamma_{jm}(h)\, e^{-i\omega h} \tag{17}$$

is a continuous periodic function of ω called the cross-spectral function of $X_j(t)$ and $X_m(t)$, for $-\infty < \omega < \infty$ and $j, m = 1, \ldots, r$.

If we let $\underline{F}(\omega)$ be the spectral density matrix with a typical element represented by $f_{jm}(\omega)$, we can use the following matrix representation

$$\underline{\Gamma}(h) = \int_{-\pi}^{\pi} e^{i\omega h}\, \underline{F}(\omega)\, d\omega \tag{18}$$

and

$$\underline{F}(\omega) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-i\omega h}\, \underline{\Gamma}(h) \tag{19}$$

Let us look at some properties of Eq (17) following Brillinger's discussion (Ref 2:23-25). The requirement that $[\gamma_{jm}(h)]$ is absolutely summable says that $f_{jm}(\omega)$ is bounded and uniformly continuous. If the components of $\underline{X}(t)$ are real-valued then

$$f_{jm}(\omega) = f^{*}_{jm}(-\omega) = f_{mj}(-\omega) = f^{*}_{mj}(\omega) \tag{20}$$

where f^{*}_{jm} indicates the complex conjugate of f_{jm}. We can also see that $f_{jm}(\omega)$ has a period of 2π with respect to ω

since $f_{jm}(\omega) = f_{jm}(\omega + 2\pi)$. If $j = m$ then we call $f_{jj}(\omega)$ the power spectrum of the time series $X_j(t)$ at the frequency ω. If $j \neq m$ then $f_{jm}(\omega)$ is called the cross spectrum of the time series $X_j(t)$ with the time series $X_m(t)$ at the frequency ω.

To look at some properties of the spectral density matrix (Eq (19)) we need to look at two definitions from Fuller (Ref 4:154). First, a square complex valued matrix $\underline{B}$ is called a Hermitian matrix if it is equal to its conjugate transpose; i.e.,

$$\underline{B} = \underline{B}^*$$

where the jm^{th} element of $\underline{B}^*$ is the complex conjugate of b_{mj}, the mj^{th} element of $\underline{B}$.

The second definition is that a Hermitian matrix $\underline{B}$, is positive definite if for any complex vector $\underline{w}$ such that $\underline{w}^* \underline{w} > 0$,

$$\underline{w}^* \underline{B} \underline{w} > 0,$$

and it is positive semidefinite if

$$\underline{w}^* \underline{B} \underline{w} \geq 0.$$

Brillinger calls positive semidefinite non-negative definite. We will use Fuller's version of the following theorem (Ref 4:155) because Fuller has a proof, even though Brillinger also states it (Ref 2:24).

Theorem 4:

For a stationary time series of dimension r satisfying

$$\sum_{h=-\infty}^{\infty} |\gamma_{jm}(h)| < \infty$$

for j, m = 1, 2, . . . , r, the matrix $\underline{F}(\omega)$ is a positive semidefinite Hermitian matrix for all ω in $[-\pi, \pi]$.

Theorem 4 tells us that $\underline{F}^*(\omega) = \underline{F}'(\omega)$. Combining Theorem 4 and the properties of symmetry and periodicity that we have discussed above indicate that the domain of ω may be restricted to the interval $[0, \pi]$.

Now that we have looked at the spectral density matrix and its properties we need to look at how to estimate it with the realization of a time series we are working with. We will estimate f_{jm} with a second-order periodogram, I_{jj}, defined to be (Ref 2:120, 235)

$$I_{jj}(\lambda) = (2\pi T)^{-1} \sum_{t=1}^{T} \left| X_j(t)\, e^{i\lambda t} \right|^2 \qquad (21)$$

where T is the number of data points in $X_j(t)$. Expanding to the case of a vector time series we have

$$I_{jm}(\lambda) = (2\pi T)^{-1} \left| \sum_{t=1}^{T} X_j(t)\, e^{-i\lambda t} \right| \left| \sum_{t=1}^{T} X_m(t) e^{-i\lambda t} \right| \qquad (22)$$

for $\lambda \neq 0, \pm 2\pi, \ldots$. We can also represent the periodogram as a matrix

$$\underline{I}(\lambda) = [I_{jm}(\lambda)] \tag{23}$$

or

$$\underline{I}(\lambda) = (2\pi T)^{-1} \left(\sum_{t=1}^{T} \underline{X}(t)\, e^{-i\lambda t} \right) \left(\sum_{t=1}^{T} \underline{X}(t)\, e^{-i\lambda t} \right)^{*} \tag{24}$$

for $j, m = 1, \ldots, r$ and $-\infty < \lambda < \infty$. It can be seen that $\underline{I}(\lambda)$ has the same properties of symmetry and periodicity as $\underline{F}(\omega)$. The distribution of $\underline{I}$ is an important item to consider; it is given in the next theorem by Brillinger (Ref 2:238).

Theorem 5:

Let $\underline{X}(t)$, $t = 0, \pm 1, \ldots$ be an r vector-valued series satisfying Eq (5). Let $\underline{I}(\lambda)$ be defined as in Eq (24). Let $s_j(T)$ be an integer with $\lambda_j(T) = 2\, s_j(T)/T$ tending to λ_j as $T \longrightarrow \infty$ for $j = 1, \ldots, J$. Suppose $2\,\lambda_j(T)$, $\lambda_j(T) \pm \lambda_k(T) \not\equiv 0 \pmod{2\pi}$ for $1 \leq j < k \leq J$ and T sufficiently large. Then $\underline{I}(\lambda_j(T))$, $j = 1, \ldots, J$ are asymptotically independent $W_r^C(1, \underline{F}(\lambda_j))$, $j = 1, \ldots, J$. Also if $\lambda = \pm\pi, \pm 3\pi, \ldots$, $\underline{I}(\lambda)$ is asymptotically $W_r(1, \underline{F}(\lambda))$ independently of the previous variates.

As a note, the notation $W_r^C(df, \underline{F}(\omega))$ indicates a central

complex Wishart distribution of dimension r and df degrees of freedom with a covariance matrix of $\underline{F}(\omega)$. The W_r without the superscript C indicates a Wishart distribution of dimension r. Note that W_r is a special case of W_r^C.

An important outcome of this theorem is that when $\lambda_j = \lambda$, we have a source for J asymptotically independent estimates of $\underline{F}(\lambda)$. The theorem also points out a limitation of using Eq (24) as estimate. With one degree of freedom, the Wishart distribution is well spread out about $\underline{F}(\lambda)$. Brillinger (Ref 2:240) arrives at the conclusion "that second-order periodograms were not reasonable estimates of second order spectra." The main reason is that $\underline{I}$ is not consistent, that is, the estimate does not improve as T gets larger. We can increase the degrees of freedom of the complex Wishart distribution by averaging a certain number of asymptotically independent variates together. The higher degrees of freedom would lessen the spread of the distribution around $\underline{F}(\lambda)$. This idea leads us to the smoothed periodogram which we will discuss next.

Krishnaiah (Ref 6:20) recommends the following estimate for $\underline{F}(\omega)$

$$\hat{\underline{F}}(\omega) = [\hat{f}_{jk}(\omega)] \tag{25}$$

$$\hat{f}_{jk}(\omega) = \sum_{a=-m}^{m} W_a \, I_{jk}\left(\omega + \frac{2\pi a}{T}\right) \tag{26}$$

with

$$\lambda = \omega + \frac{2\pi a}{T} \tag{27}$$

$$I_{jk}(\lambda) = Z_j(\lambda)\, Z_k^*(\lambda) \tag{28}$$

and

$$Z_j(\lambda) = 1/(2\pi T)^{\frac{1}{2}} \sum_{t=1}^{T} X_j(t)\, e^{-i\omega t} \tag{29}$$

where W_a are weights whose sum equals one and T is the number of data points in the scalar time series $X_j(t)$. In this representation $I_{jk}(\lambda)$ is a second order periodogram and $\hat{f}_{jk}(\omega)$ is the average of (2m + 1) periodograms and is called the smoothed periodogram.

Brillinger (Ref 2:242-3) recommends the following estimates with $I_T(\omega)$ given as thus

$$I_T(\omega) = (2\pi T)^{-1} \left(\sum_{t=1}^{T} \underline{X}(t)\, e^{-i\omega t} \right) \left(\sum_{t=1}^{T} \underline{X}(t)\, e^{-i\omega t} \right)^* \tag{30}$$

$$= \underline{Z}\, \underline{Z}^* \tag{31}$$

which is the same as Krishnaiah. However, Brillinger gives three estimates for $\hat{\underline{F}}(\omega)$ depending on the value of (ω).

$$\hat{\underline{F}}(\omega) = (2m + 1)^{-1} \sum_{a=-m}^{m} \underline{I}\left(\frac{2\pi[s(T) + a]}{T} \right) \tag{32}$$

$$\text{if } \omega \not\equiv 0 (\text{mod}\pi)$$

$$\hat{\underline{F}}(\omega) = m^{-1} \sum_{a=1}^{m} \text{Real}\left(\underline{I}\left(\omega + \frac{2\pi a}{T}\right)\right) \qquad (33)$$

if $\omega = 0, \pm 2\pi, \ldots,$ or if
$\omega = \pm\pi, \pm 3\pi, \ldots$ and T is even

$$\hat{\underline{F}}(\omega) = m^{-1} \sum_{a=1}^{m} \text{Real}\left(\underline{I}\left(\omega - \frac{\pi}{T} + \frac{2\pi a}{T}\right)\right) \qquad (34)$$

if $\omega = \pm\pi, \pm 3\pi, \ldots$ and T is odd

where $2\pi s(T)/T \longrightarrow \omega$ as $T \longrightarrow \infty$. These estimates have the same properties of symmetry and periodicity as $\underline{F}(\omega)$ in Eq (19). This estimate is asymptotically unbiased as indicated in this theorem from Brillinger (Ref 2:244).

Theorem 6:

Let $\underline{X}(t)$, $t = 0, \pm 1, \ldots$ be an r vector-valued series with mean function $\underline{\mu}$ and cross-variance function $\underline{\Gamma}(h)$ defined in Eq (4). Suppose $\underline{\Gamma}(h)$ is absolutely summable as defined in Eq (5). Let $\hat{\underline{F}}(\omega)$ be given by Eqs (32), (33) and (34) and if $2\pi s(T)/T \longrightarrow \omega$ as $T \longrightarrow \infty$ then

$$\lim_{T \longrightarrow \infty} E(\underline{F}(\omega)) = \underline{F}(\omega) \qquad \text{for } -\infty < \omega < \infty$$

Once we have an estimate of $\underline{F}(\omega)$ we need to look at its distribution. Again Brillinger (Ref 2:245) gives us a theorem on the distribution of $\hat{\underline{F}}(\omega)$.

Theorem 7:

Let $\underline{X}(t)$, $t = 0, \pm 1, \ldots$ be an r vector-valued series satisfying Eq (5). Let $\hat{\underline{F}}(\omega)$ be given by Eqs (32), (33), and (34) with $2\pi s(T)/T \longrightarrow \omega$ as $T \longrightarrow \infty$. Then $\hat{\underline{F}}(\omega)$ is asymptotically distributed as $(2m + 1)^{-1} C_r (2m + 1, \underline{F}(\omega))$ if $\omega \not\equiv 0 (\mathrm{mod}\pi)$ and as $(2m)^{-1} W_r (2m, \underline{F}(\omega))$ if $\omega \equiv 0(\mathrm{mod}\pi)$. Also $\hat{\underline{F}}(\omega j)$, $j = 1, \ldots, J$ are asymptotically independent if $\lambda_j \pm \lambda_k \not\equiv 0 (\mathrm{mod}\ 2\pi)$ for $1 \leq j < k \leq J$.

Thus our current estimators are "better" (relatively speaking) than a non-smoothed periodogram.

In Chapter III, we will discuss how we tested this procedure. At this time we will look at a theoretical calculation of a spectral density matrix and a confidence interval on the estimated spectral density matrix. Our vector time series are three first order autoregressive (AR(1)) scalar stationary time series with the covariance between any two zero so we can look at each scalar time series separately. Fuller develops the theoretical spectral density for this process (Ref 4:143-144). First, we need a theorem about the spectral density matrix of this particular process.

Theorem 8 (Ref 4:140-141):

Let X(t) be a stationary time series with an absolutely summable covariance function and let $\{a_j\}_{j=-\infty}^{\infty}$ be absolutely summable. Then the spectral density of

$Y(t) = \sum_{j=-\infty}^{\infty} a_j X_{t-j}$ is

$$f_Y(\omega) = (2\pi)^2 f_X(\omega) f_a(\omega) f_a^*(\omega) \tag{35}$$

where $f_X(\omega)$ is the spectral density of X(t),

$$f_a(\omega) = (2\pi)^{-1} \sum_{j=-\infty}^{\infty} a_j e^{-i\omega j} \tag{36}$$

is the Fourier transform of a_j, and

$$f_a^*(\omega) = (2\pi)^{-1} \sum_{j=-\infty}^{\infty} a_j e^{i\omega j} \tag{37}$$

is the complex conjugate of the Fourier transform of a_j.

For this development we follow Fuller (Ref 4:143-144). A first order autoregressive process can be written as

$$X(t) = \sum_{j=0}^{\infty} \rho^j e_{t-j} \tag{38}$$

By Theorem 8 the transform of the weights $a_j = \rho^j$ is given by Eq (35)

$$g_a(\omega) = \frac{1}{2\pi} \sum_{j=0}^{\infty} \rho^j e^{-ij\omega}$$

$$= \frac{1}{2\pi} \sum_{j=0}^{\infty} (\rho e^{-i\omega})^j$$

$$= \frac{1}{2\pi} \frac{1}{1 - \rho \exp(-i\omega)}$$

and its complex conjugate is, from Eq (37),

$$g_a^*(\omega) = \frac{1}{2\pi} (\rho^j e^{ij\omega})$$

$$= \frac{1}{2\pi} \frac{1}{1 - \rho \exp(i\omega)}$$

Hence, also by Theorem 8 (Eq (35)) we have

$$f_x(\omega) = (2\pi)^2 f_e(\omega) g_a(\omega) g_a^*(\omega)$$

$$= \frac{g^2}{2\pi} \left| \frac{1}{1 - \rho \exp(-i\omega)} \right| \left| \frac{1}{1 - \rho \exp(i\omega)} \right|$$

$$= \frac{g^2}{2\pi} \left| \frac{1}{1 - 2\rho \cos\omega + \rho^2} \right| \qquad (39)$$

since $f_e(\omega)$ is the spectral density of the uncorrelated sequence $\{e_t\}$ and equals $\sigma^2/2\pi$ for all ω.

To get an approximate $(1 - \alpha)$ significance level confidence interval we will again use Fuller's estimate. Fuller summarizes a long discussion (Ref 4:287-295) into the following result (Ref 4:296) which we will use. We let X(t) be an autoregressive of order 1 that can be represented as an infinite moving average of the form

$$X(t) = \sum_{j=0}^{\infty} \alpha_j e_{t-j}$$

where $\alpha_j = p^j$ and the e_t are independent $(0, \sigma^2)$ random variables and let $f(\omega) > 0$. Then, for $\frac{\pi m}{T} < \omega < \pi(1 - \frac{m}{T})$, we have $(f^{-1}(\omega)\ \hat{f}(\omega))$ is approximately distributed as a chi-square random variable divided by its degrees of freedom ν, where

$$\nu = 2\left(\sum_{j=-m}^{m} W^2(j)\right)^{-1} \tag{40}$$

and where $\hat{f}(\omega)$ is the smoothed periodogram, the average of $(2m + 1)$ periodograms. $W(j)$ is a weight such that

$$\sum_{j=-m}^{m} W(j) = 1 \tag{41}$$

$$W(j) = W(-j) \tag{42}$$

We can construct the $(1 - \alpha)$ level confidence interval for $f(\omega)$ to be

$$\frac{\nu\ \hat{f}(\omega)}{\chi^2_{\nu,\ \alpha/2}} \leq f(\omega) \leq \frac{\nu\ \hat{f}(\omega)}{\chi^2_{\nu,\ 1 - (\alpha/2)}} \tag{43}$$

where $\chi^2_{\nu,\ \alpha/2}$ is the $\alpha/2$ tabular value for the chi-square distribution with ν degrees of freedom.

This estimator is for the diagonal elements of the spectral density matrix. For the cross spectral elements we can find a confidence interval for the amplitude of the spectrum (Ref 4:156, 314-317). Let the cross spectral

estimate be represented as

$$\hat{f}_{jk} = \hat{c}_{jk} + i\,\hat{q}_{jk} \tag{44}$$

then the amplitude of f_{jk}, A_{jk}, can be estimated by

$$\hat{A}_{jk} = \left(\hat{c}^2_{jk} + \hat{q}^2_{jk}\right)^{\frac{1}{2}} \tag{45}$$

Fuller does not state a theorem but he says that we can assume $\hat{c}_{jk}$ and $\hat{q}_{jk}$ to be normally distributed (Ref 4:316-317). With this assumption we can get an upper and lower bound on A_{jk}.

The upper bound can be calculated as

$$A^U_{jk}(\omega) = \hat{A}_{jk}(\omega) + \left[(2m+1)^{-1}\,\hat{f}_{jj}(\omega)\,\bar{f}_{kk}(\omega)\,F^2_{4m}(\alpha)\right]^{\frac{1}{2}} \tag{46}$$

and

$$A^L_{jk}(\omega) = \max\left\{0,\ \hat{A}_{jk}(\omega) - \left[(2m+1)^{-1}\,\hat{f}_{jj}(\omega)\bar{f}_{kk}(\omega)F^2_{4m}(\alpha)\right]^{\frac{1}{2}}\right\} \tag{47}$$

where

$$\bar{f}_{kk}(\omega) = \hat{f}_{kk}\left|1 - \hat{K}^2_{jk}(\omega)\right|\left(\frac{2m+1}{2m}\right)$$

$$\hat{K}^2_{jk} = \frac{|\hat{f}_{jk}(\omega)|^2}{\hat{f}_{jj}(\omega)\,\hat{f}_{kk}(\omega)} = \frac{\hat{c}^2_{jk}(\omega) + \hat{q}^2_{jk}(\omega)}{\hat{f}_{jj}(\omega)\,\hat{f}_{kk}(\omega)}$$

and where F^2_{4m} is Snedecor's F with 2 and 4m degrees of freedom. Hence, combining Eqs (46) and (47) we have

$$A^L_{jk}(\omega) \leq A(\omega) \leq A^U_{jk}(\omega) \tag{48}$$

Note that $A^U_{jk} = A^U_{kj}$ and $A^L_{jk} = A^L_{kj}$.

The Test

This completes the theory we need to get into the procedure we will be using. As stated earlier, our procedure is based on the work of P. R. Krishnaiah and F. J. Schuurmann. Since we will be accomplishing two tests, a Beta test and an F test, we will be discussing two test statistics. We will look at the test statistics first. Kirshnaiah and Shuurmann state

> Let S_1 and S_2 be distributed independently as central complex Wishart matrices with m and n degrees of freedom, respectively, and let $E(S_1/m) = E(S_2/n) = I_p$, where I_p is the identity matrix. Then, $S_1(S_1 + S_2)^{-1}$ is known to be a central complex multivariate beta matrix, whereas $S_1 S_2^{-1}$ is known to be a central complex multivariate F matrix (Ref 7:2).

Let $\{\underline{X}(t), t = 0, 1, \ldots\}$ and $\{\underline{Y}(t), t = 0, 1, \ldots\}$ be r-variate Gaussian time series with $E(\underline{X}(t)) = \underline{0}$ and $E(\underline{Y}(t)) = \underline{0}$ for all t and let $\underline{X}(t)$ and $\underline{Y}(t)$ be covariance stationary with the elements of their respective covariance matrices absolutely summable. Let $\underline{F}(\omega)$ be the spectral density matrix of $\underline{X}(t)$ multiplied by its degrees of freedom and $\underline{G}(\omega)$ be the spectral density matrix of $\underline{Y}(t)$ multiplied by its degrees of freedom.

The null hypothesis, H, of the test is that $F(\omega_\ell) = G(\omega_\ell)$ for $\ell = 1, 2, \ldots, q$, while the alternative hypothesis is that $F(\omega_\ell) \neq G(\omega_\ell)$ for $\ell = 1, 2, \ldots, q$. We will be looking at two different tests, a Beta test and an F test. We will talk about them in parallel but not as one over the second. We will accept H if the Beta test statistic

$$\mathrm{tr}[\hat{\underline{F}}(\omega_\ell)(\hat{\underline{F}}(\omega_\ell) + \hat{\underline{G}}(\omega_\ell))^{-1}] \leq a_\alpha \qquad (49)$$

where tr means trace and a_α is the table value described below. Similarly we accept H if the F test statistic

$$\mathrm{tr}[\hat{\underline{F}}(\omega_\ell)(\underline{G}(\omega_\ell))^{-1}] \leq b_\alpha \qquad (50)$$

where b_α is the table value described below. We do not require that both tests pass to accept H. We only require that the test we are looking at pass.

The table value a_α is the upper percentage point of the distribution of the trace of the complex multivariate beta matrix where α indicates the probability of making a type 1 error. These values are found in (Ref 7:12-63). The table value b_α is the upper percentage point of the distribution of the trace of the complex multivariate F matrix, with α, again, indicating the probability of making a type 1 error. These values are also found in (Ref 6:64-99). The

a_α and b_α are chosen such that

$$\prod_{\ell=1}^{q} P[\mathrm{tr}\{\hat{\underline{F}}(\omega_\ell)(\hat{\underline{F}}(\omega_\ell) + \underline{G}(\omega_\ell))^{-1}\} \leq a_\alpha | H] = (1 - \alpha) \quad (51)$$

and

$$\prod_{\ell=1}^{q} P[\mathrm{tr}\{\underline{F}(\omega_\ell)\ \underline{G}(\omega_\ell)^{-1}\} \leq b_\alpha | H] = (1 - \alpha) \quad (52)$$

Our discussion has been for a one-tail test. If you want to accomplish a two-tail test just replace alpha by $(\alpha/2)$ in Eqs (51) and (52).

In this chapter we have looked at the notation and terminology we used and we took a brief look at the theory of using the frequency domain to analyze time series. We then discussed the procedure and the test statistics we used. In the next chapter we will discuss the simulations we did and the results we got from them.

III. Monte Carlo Simulations

This chapter discusses the Monte Carlo simulations accomplished to test the procedure against randomly generated time series. As stated in Chapter II, the scalar time series are first order autoregressive stationary time series of the form

$$X_r(t) = e(t) \qquad t=1$$

$$X_r(t) = \rho_r X_r(t-1) + e(t) \qquad t=2,\ldots, T \qquad (53)$$

where T is the number of data points. The e(t) are normal (0,1) random deviates for t=2,..., T. For t=1, e(t) was adjusted to incorporate the effects of the terms between $-\infty$ and 0 which ensures that the process is stationary. The absolute value of each ρ_r was less than one. The e(t) sequences were calculated as follows. Using the International Mathematical and Statistical Libraries, Inc. (IMSL) routine GGUB, we obtained a sequence of uniform (0,1) deviates (Ref 5:GGUB-1). The deviates were converted to normal (0,1) deviates by using the Box-Mueller technique. Three scalar time series were generated using this procedure to form one $\underline{X}(t)$. The cross covariance between $X_j(t)$ and $X_m(t)$ is zero for $j \neq m$ so $\underline{X}(t)$ is a stationary vector time series.

The spectral density matrix was calculated and both test statistics were calculated. The test of hypothesis used was a two-tail test at four different alpha values, 0.2, 0.1, 0.05, and 0.02. In the first two simulations we calculated the test statistic (using the notation of the procedure discussion in Chapter II) as

$$\mathrm{tr}\{\hat{\underline{F}}(\omega_\ell)[\hat{\underline{F}}(\omega_\ell) + \hat{\underline{G}}(\omega_\ell)]^{-1}\} = \mathrm{TR1} \tag{54}$$

and

$$\mathrm{tr}\{\hat{\underline{G}}(\omega_\ell)[\hat{\underline{F}}(\omega_\ell) + \hat{\underline{G}}(\omega_\ell)]^{-1}\} = \mathrm{TR2} \tag{55}$$

for the Beta test and as

$$\mathrm{tr}[\hat{\underline{F}}(\omega_\ell)(\hat{\underline{G}}(\omega_\ell))^{-1}] = \mathrm{TR3} \tag{56}$$

and

$$\mathrm{tr}[\hat{\underline{G}}(\omega_\ell)(\hat{\underline{F}}(\omega_\ell))^{-1}] = \mathrm{TR4} \tag{57}$$

for the F test.

For both tests we took the greater of TR1 and TR2 and the greater of TR3 and TR4 and tested against the respective table values. $\hat{\underline{F}}$ and $\hat{\underline{G}}$ were written using Brillinger's estimates, Eqs (32), (33), and (34) from Chapter II.

In all simulations, the number of data points, T, was 200. We restricted T to be even so we did not use Eq (33). We also set s(T) in Eq (32) to vary from 0 to 100 in intervals

of 10. This allowed us to look at eleven frequencies between zero and π, inclusively. The frequencies are $\pi/10$ apart.

The analysis can be accomplished two ways. First, using the criteria that all frequencies pass the test or using a binomial distribution. A confidence interval was determined from a table of confidence intervals for proportions. We did this because we are looking at failures or successes, a zero-one test. The alpha is the probability of failure for simulation I, at a specific frequency.

Before we look at the results of the simulations and analyze them, a couple points need to be emphasized. The sample size of each simulation is small. The first two only consist of a hundred runs each, when they should be at least 1,000 to 5,000 each. The third simulation was done with 12 runs which seems to be adequate as we will point out later. With this in mind we can still make inferences about the results, though.

The runs were limited because the program took a long time to run, about 94 seconds for each run. This length of time lowered the priority, which was already low so turn around time was long. A normal sample of 25 runs, at the minimum had an overnight turn around, though the usual was two to three days. Programs which ran only one run had an average turn around time of three hours. A second limiting factor was budget limitations. A final point we wish to

emphasize is that all the data is simulated. We were unable to receive actual data. This makes the analysis aesthetically less satisfying.

Keeping the limited number of runs in mind we will look at the first simulation. This simulation simulated two time series with equal respective rho values. The values used were -0.7, 0.3, and 0.8. Figure 1 shows the theoretical spectral density matrix for these two time series, based on Eq (39) with $\sigma^2 = 1$.

$$f_x(\omega) = \frac{1}{2\pi(1 + \rho^2 - 2\rho \cos\omega)}$$

In testing these two time series the values of TR1 from Eq (54) and TR2 from (55) should be 1.5 for all frequencies. The values of TR3 and TR4 from Eqs (56) and (57) should be 3.0 for all frequencies.

The results of this simulation are interesting. Table I shows the results at each of the alpha values for the Beta test. Table II has the same information for the F test. Both tables show that the number of failures at two frequencies, zero and π, are extremely high. We cannot explain this. The other frequencies include their respective alpha within their 95% confidence interval (see Ref 1:220). Table III shows the number of runs which failed exactly at the number of indicated failures. These can be compared with a binomial distribution (Ref 1:184) and the comparison shows

$\omega = 0$

$$\begin{bmatrix} 0.0551 & 0 & 0 \\ 0 & 0.3248 & 0 \\ 0 & 0 & 3.9789 \end{bmatrix}$$

$\omega = \pi/2$

$$\begin{bmatrix} 0.1068 & 0 & 0 \\ 0 & 0.1460 & 0 \\ 0 & 0 & 0.970 \end{bmatrix}$$

$\omega = \pi$

$$\begin{bmatrix} 1.7684 & 0 & 0 \\ 0 & 0.0942 & 0 \\ 0 & 0 & 0.0491 \end{bmatrix}$$

Figure 1. Theoretical Spectral Density Matrix, Simulation I (Selected Frequencies)

the same structure even though the table values do not match. A simulation of 1,000 or more runs should show the values converging to the table values for the binomial distribution.

There are two ways to interpret this data. From our discussion of the procedure in Chapter II we can require all tests to pass. The alpha for each test must be small to get a larger simultaneous alpha value. For example, when each run is tested with an alpha of 0.02 we see that 32 percent of the runs failed so the empirical alpha is 32 percent for this test.

Table I

Number of Failures
Beta Test, Simulation I

	Frequency										
Alpha	0	$\frac{\pi}{10}$	$\frac{\pi}{5}$	$\frac{3\pi}{10}$	$\frac{2\pi}{5}$	$\frac{\pi}{2}$	$\frac{3\pi}{5}$	$\frac{7\pi}{10}$	$\frac{4\pi}{5}$	$\frac{9\pi}{10}$	π
0.2	36	16	24	17	23	14	20	14	24	17	39
0.1	27	9	15	7	9	9	12	9	15	10	33
0.05	19	6	8	2	2	7	4	2	10	5	24
0.02	10	3	4	0	1	2	2	0	3	2	12

Table II

Number of Failures
F Test, Simulation I

	Frequency										
Alpha	0	$\frac{\pi}{10}$	$\frac{\pi}{5}$	$\frac{3\pi}{10}$	$\frac{2\pi}{5}$	$\frac{\pi}{2}$	$\frac{3\pi}{5}$	$\frac{7\pi}{10}$	$\frac{4\pi}{5}$	$\frac{9\pi}{10}$	π
0.2	78	19	26	14	17	23	20	15	25	17	77
0.1	62	13	14	7	8	10	10	5	15	6	65
0.05	50	6	9	4	3	4	8	3	9	3	59
0.02	39	0	3	1	0	2	4	0	2	3	44

Table III

Exact Number of Failures, Simulation I

Number of Failures	$\alpha = 0.2$		$\alpha = 0.1$		$\alpha = 0.5$		$\alpha = 0.02$		Number of Failures
	B Test	F Test	B Test	F Test	B Test	F Test	B Test	F Test	
0	5	2	23	5	39	18	68	31	0
1	20	6	31	20	35	37	26	44	1
2	36	15	25	39	24	31	5	21	2
3	13	34	15	28	2	12	1	4	3
4	19	27	4	7	0	2	0	0	4
5	5	12	2	1	0	0	0	0	5
6	1	4	0	0	0	0	0	0	6
7	1	0	0	0	0	0	0	0	7
8	0	0	0	0	0	0	0	0	8
9	0	0	0	0	0	0	0	0	9
10	0	0	0	0	0	0	0	0	10
11	0	0	0	0	0	0	0	0	11

An alternative method is to use the Bernoulli trial concept for independent trials and use a decision rule based on allowing a certain number of failures. In this method we set the simultaneous alpha based on the cumulative binomial distribution function. Table III and the binomial distribution show that we can expect some frequencies to fail even when the two time series are generated by the same process. Using Table III and the cumulative binomial distribution table (Ref 1:196), we determined the following decision rules, one for each alpha. At $\alpha = 0.02$, if the number of frequencies that fail is 4 or more, then the test fails and the null hypothesis is rejected. At $\alpha = 0.05$, the cut-off is 3 or more frequencies. At $\alpha = 0.1$, the cut-off is also 3 or more, at $\alpha = 0.2$, 4 or more frequencies. These cut-off values give a simultaneous alpha value different than the one being considered for each test. The value for the alpha of 0.02 was determined from Table III. For the Beta test the alpha would be about .01 and for the F test it would be .04. For the other cut-off values the true alpha would be 0.02, 0.09, and 0.16 for 0.05, 0.1 and 0.2 respectively. Table IV summarizes the cut-off values and the upper percentage points we used in the test.

An important parameter in considering cut-off values is the number of frequencies looked at. This analysis is for eleven frequencies. A different number of frequencies would require a different cut-off value. The analyst should

Table IV

Test Cut-Off Values and Upper Percentage Points Used With Eqs (49) and (50)

Alpha	Cut-off Value	a_α $\omega \equiv 0$ (modπ)	a_α $\omega \not\equiv 0$ (modπ)	b_α $\omega \equiv 0$ (modπ)	b_α $\omega \not\equiv 0$ (modπ)
0.20	4	1.715	1.705	5.516	5.299
0.10	3	1.77	1.763	6.173	5.889
0.05	3	1.823	1.813	6.83	6.474
0.02	4	1.88	1.870	7.72	7.256

use a cumulative distribution table and choose the value such that the probability of X or more failures is the value of the alpha level he requires. An alternative procedure requiring more calculations is to use Bonferroni's inequality (Ref 3:100).

The second simulation used one time series with the same parameters as the first simulation. The first time series has rho values of 0.2, -0.5, and -0.9. It was felt that these rho values were far enough apart from the rho values for the other time series to ensure that the null hypothesis would be rejected. When the theoretical spectral density matrix for the different time series was calculated (see Figure 2) and the test statistics were calculated using Eqs (54), (55), (56), and (57), a surprising thing was noted. Some of the frequencies would pass the test of hypothesis

$\omega = 0$

$$\begin{bmatrix} 0.2487 & 0 & 0 \\ 0 & 0.0707 & 0 \\ 0 & 0 & 0.441 \end{bmatrix}$$

$\omega = \pi/2$

$$\begin{bmatrix} 0.1530 & 0 & 0 \\ 0 & 0.1273 & 0 \\ 0 & 0 & 0.0879 \end{bmatrix}$$

$\omega = \pi$

$$\begin{bmatrix} 0.1105 & 0 & 0 \\ 0 & 0.6366 & 0 \\ 0 & 0 & 15.9155 \end{bmatrix}$$

Figure 2. Theoretical Spectral Density Matrix for First Time Series, Simulation II (Selected Frequencies)

as being equal. Table V shows the traces calculated by Eqs (54) and (55) for the Beta test. At ω_4 through ω_9, depending on the level of confidence, we expect these frequencies to not reject the null hypothesis. Similarly, Table VI shows the same type of information for the F test.

The simulation agreed with this theoretical analysis. Table VII shows, for the Beta test, the number of failures for each of the alphas we looked at and Table VIII shows the same information for the F test. In Table VII, the alpha characters below the numbers indicate whether or not that

Table V

Theoretical Traces for Beta Test, Simulation II

ω_i	TR1	TR2	α at which test passes
1	1.0151	1.9916	Fail
2	1.0340	1.9677	Fail
3	1.1078	1.8922	Fail
4	1.2243	1.7757	0.05
5	1.3709	1.6291	0.20
6	1.5301	1.4699	0.20
7	1.6814	1.3186	0.2
8	1.8030	1.197	0.05
9	1.8618	1.1185	0.02
10	1.9178	1.0823	Fail
11	1.9262	1.0731	Fail

Table VI

Theoretical Traces for F Test, Simulation II

ω_i	TR3	TR4	α at which test passes
1	4.7424	95.0400	Fail
2	4.548	34.2282	Fail
3	4.0596	13.1404	Fail
4	3.5201	6.9400	0.02
5	3.1627	4.2634	0.2
6	2.4260	2.9484	0.2
7	3.9651	2.4260	0.2
8	6.1311	2.6625	0.05
9	12.1361	4.162	Fail
10	37.8965	9.1714	Fail
11	330.8203	16.1548	Fail

Table VII

Number of Failures
Beta Test, Simulation II

	Frequency										
Alpha	0	$\frac{\pi}{10}$	$\frac{\pi}{5}$	$\frac{3\pi}{10}$	$\frac{2\pi}{5}$	$\frac{\pi}{2}$	$\frac{3\pi}{5}$	$\frac{7\pi}{10}$	$\frac{4\pi}{5}$	$\frac{9\pi}{10}$	π
0.20	95	100	98	60	16	17	34	73	94	100	95
	F	F	F	F	P	P	P	F	F	F	F
0.10	87	96	89	49	7	8	24	59	86	95	93
	F	F	F	F	P	P	P	F	F	F	F
0.05	78	92	78	30	4	3	12	42	67	91	85
	F	F	F	P	P	P	P	P	F	F	F
0.02	63	79	51	14	0	2	7	29	43	70	62
	F	F	F	P	P	P	P	P	P	F	F

NOTE: F = Frequency theoretically Failed
P = Frequency theoretically Passed

Table VIII

Number of Failures
F Test, Simulation II

	Frequency										
Alpha	0	$\frac{\pi}{10}$	$\frac{\pi}{5}$	$\frac{3\pi}{10}$	$\frac{2\pi}{5}$	$\frac{\pi}{2}$	$\frac{3\pi}{5}$	$\frac{7\pi}{10}$	$\frac{4\pi}{5}$	$\frac{9\pi}{10}$	π
0.20	100	100	100	98	54	26	33	93	100	100	100
	F	F	F	F	P	P	P	F	F	F	F
0.10	100	100	100	89	32	13	22	82	100	100	100
	F	F	F	F	P	P	P	F	F	F	F
0.05	100	100	100	83	20	5	14	76	100	100	100
	F	F	F	F	P	P	P	P	F	F	F
0.02	100	100	99	73	12	2	6	63	99	100	100
	F	F	F	P	P	P	P	P	F	F	F

NOTE: F = Frequency theoretically Failed
P = Frequency theoretically Passed

that frequency was supposed to pass. In this test each alpha had at least one frequency where the number of failures did not have the alpha value in its 95 percent confidence interval. Omega of $3\pi/5$ failed more times than expected, except for alpha equal to 0.05 the number of failures did not include the alpha level in a 99 percent confidence interval. Only two omega values passed as expected, they are omega $2\pi/5$ and $\pi/2$. Again, we do not have a reason for this, however, a simulation of a larger number of runs may help to bring the values down.

Table VIII shows a similar structure for the F test results. Only omega $\pi/2$ shows values that include the alpha in a 95 percent or 99 percent confidence interval. All the other frequencies that passed do not. Again, we do not have a reason for this, other than this is a result of the small sample size.

Table IX shows the number of runs which had the indicated number of failures. Using the binomial distribution decision criteria developed for simulation I (Table IV), we would reject the null hypothesis that these two time series came from the same process. The only problem is at alpha equal to 0.02 where 13 runs passed with less than four frequency failures.

The first two simulations show that we can accept or reject the null hypothesis for simulated time series. The problem that was mentioned during the discussion of

Table IX

Exact Number of Failures, Simulation II

Number of Failures	$\alpha = 0.2$		$\alpha = 0.1$		$\alpha = 0.05$		$\alpha = 0.02$		Number of Failures
	B Test	F Test	B Test	F Test	B Test	F Test	B Test	F Test	
0	0	0	0	0	0	0	2	0	0
1	0	0	0	0	0	0.	1	0	1
2	0	0	0	0	0	0	10	0	2
3	0	0	0	0	3	0	20	0	3
4	0	0	4	0	20	0	26	0	4
5	3	0	10	0	16	0	19	1	5
6	7	0	23	1	24	5	15	9	6
7	29	3	27	18	28	25	6	41	7
8	34	29	25	36	8	43	1	34	8
9	20	38	10	35	1	22	0	15	9
10	7	21	1	7	0	5	0	0	10
11	0	9	0	3	0	0	0	0	11

simulation I caused us to question the estimator for $\omega = 0$ and $\omega = \pi$, Eq (33). To look at the estimators we accomplished simulation III to calculate the spectral density matrix of a time series. The rho values we chose were the same as the ones for simulation I which were -0.7, 0.3, and 0.8. After averaging 12 time series, the difference between the average of 10 and 12 time series was only in the third decimal place, so we can assume the process has settled down and that a sample size of 12 is adequate for this analysis.

Terms on the order of magnitude of 10^{-14} or less have been set to zero because 10^{-14} is the round-off error for the CDC 6600 computer the simulation was run on. A 95 percent confidence interval using Eq (43) for the diagonal elements and Eq (48) for the off diagonal elements. Figure 3 shows the spectral density matrix for the three values of omega and Figure 4 shows the 95 percent confidence interval for these values. The values in Figure 1 are included in the 95 percent confidence intervals for the estimates.

Note that the actual values for omega equal to zero and π are within the 95 percent confidence interval for the estimates. This, then, cannot explain the bad results we had at those frequencies.

Hence, we are reasonably sure that our estimates are estimating the actual spectral density matrix. We can also conclude that the results of simulations I and II are feasible, keeping in mind the small sample size.

$\omega = 0$

$$\begin{bmatrix} (0.0591, 0) & (0.0116, 0.0129) & (-0.0072, 0.0130) \\ (0.0116, -0.0129) & (0.3220, 0) & (0.0866, 0.0179) \\ (-0.0072, -0.0130) & (0.0866, -0.0179) & (3.1059, 0) \end{bmatrix}$$

$\omega = \pi/2$

$$\begin{bmatrix} (0.1180, 0) & (-0.0022, -0.0016) & (0.0001, -0.0031) \\ (-0.0022, 0.0016) & (0.1403, 0) & (0.0022, -0.0010) \\ (0.0001, 0.0031) & (0.0022, 0.0010) & (0.1067, 0) \end{bmatrix}$$

$\omega = \pi$

$$\begin{bmatrix} (1.5456, 0) & (-0.0113, 0) & (-0.0119, 0) \\ (-0.0113, 0) & (0.1119, 0) & (-0.0003, 0) \\ (-0.0119, 0) & (-0.0003, 0) & (0.0451, 0) \end{bmatrix}$$

NOTE: Each number is a complex number where (a,b) means a + bi.

Figure 3. Spectral Density Matrix, Simulation III (Selected Frequencies)

$\omega = 0$

$$\begin{bmatrix} (0.0389,0.1024) & (0,0.0901) & (0,0.2428) \\ (0,0.0901) & (0.2122,0.5580) & (0,0.6156) \\ (0,0.2428) & (0,0.6186) & (2.0467,5.3821) \end{bmatrix}$$

$\omega = \pi/2$

$$\begin{bmatrix} (0.0789,0.1995) & (0,0.0659) & (0,0.0628) \\ (0,0.0659) & (0.0938,0.2372) & (0,0.2273) \\ (0,0.0628) & (0,0.2273) & (0.0713,0.1804) \end{bmatrix}$$

$\omega = \pi$

$$\begin{bmatrix} (1.0185,2.6783) & (0,0.2253) & (0,0.1521) \\ (0,0.2253) & (0.0738,0.1941) & (0,0.0487) \\ (0,0.1521) & (0,0.0487) & (0.0297,0.0782) \end{bmatrix}$$

The diagonal elements are for actual values of the power spectrum, while off diagram elements are for the amplitude of the cross spectrum.

Figure 4. 95 Percent Confidence Intervals, Simulation III

IV. Conclusions and Recommendations

We took P. R. Krishnaiah and F. J. Schuurmann's theoretical work and showed the development of the theory and applied it to some simulations. We cannot make definitive conclusions based on the simulations because the sample size was too small. We did see that some problems exist at the frequencies zero and π which cannot be explained by theory. A larger sample size is needed to make quantitative judgements on that aspect.

Based on the development of the theory there is no reason that this technique will not work for a r-variate vector time series. We did not restrict the theory to the three variable vector-valued time series that we did our simulations on. In the same vein, there is no reason that this technique will not work for processes other than the first order autoregressive process we simulated.

The computer program we used is not attached to this thesis because there are several Fast Fourier Transformation programs available that can be used to calculate the spectral density matrix. When applying this technique to AFTECs problem or any similar problem, analysts can use the canned programs to set up their spectral density matrix and easily multiply them by the degrees of freedom and calculate the test statistics, Eqs (54) through (57).

An important consideration that AFTEC or other users of this technique should be aware of is the selection of the number of frequencies tested. They should select enough frequencies far enough apart so that the probability that all frequencies pass is small. This causes the probability of a type II error to be small. The analyst controls the probability of a type I error. The frequencies looked at should be harmonics or every other harmonic or any other consistent scheme. They have to balance their desire for accuracy with the cost of examining at a greater number of frequencies. It is a common problem in testing to have to balance the probability of a type I error with the probability of a type II error.

Another area that they need to understand is model building in time series. They should be able to take the data and see what kind of process and what the parameters of that process are using least sums of squares or maximum liklihood estimators. This is especially important when they get data that rejects at several frequencies while passing at several frequencies as our simulation II did.

There are some areas where further work is needed, using this thesis as the starting point. The first is the problem with frequencies zero and π. An investigation into why these two frequencies exhibited the behavior we saw is needed. It may be just that our sample size was too small; however we cannot definitely say that is true.

A comparison of Tables II and VIII with Tables I and VII and Table III with Table IX, indicates that the F test is more sensitive than the Beta test. Again, our small sample size precludes us from making a definitive statement and it should be investigated. The problem could be in the computer rounding on cos and sin for these frequencies.

AFTEC and other users need to consider the two alternative decision rules we looked at:

(i) all pass

(ii) Binomial distribution, m or more out of n have to fail to reject.

It would seem that (i) would have a smaller probability of type II error. Further investigation of simultaneous tests to see what happens to the probability of type II error would be fruitful. The analysts will have to select the decision rule they are most comfortable with and build the test around it.

Bibliography

1. Beyer, William H. (Editor). Handbook of Tables for Probability and Statistics (Second edition). The Chemical Rubber Co., Cleveland, 1968.

2. Brillinger, David R. Time Series Data Analysis and Theory. Holt, Rinehart and Winston, Inc., New York, 1975.

3. Feller, William. An Introduction to Probability Theory and Its Applications (Second edition). John Wiley and Sons, Inc., New York, 1957.

4. Fuller, Wayne A. Introduction to Statistical Time Series. John Wiley and Sons, Inc., New York, 1976.

5. International Mathematical and Statistical Libraries, Inc. IMSL Library 3 Reference Manual. Houston, 1977.

6. Krishnaiah, P. R. "Some Recent Developments on Complex Multivariate Distributions," Journal of Multivariate Analysis, 6, 1-30 (1976).

7. Krishnaiah, P. R. and F. J. Schuurmann. Approximations to the Distributions of the Traces of Complex Multivariate Beta and F Matrices. ARL 74-0123 (AD A006 153). Aerospace Research Laboratories, Wright-Patterson Air Force Base, Ohio, 1974.

VITA

Anthony L. Bertapelle graduated from Adams City High School, Commerce City, Colorado, in 1965. He attended the University of Northern Colorado, graduating in 1969 with a Bachelor of Art in mathematics. He received his commission in the United States Air Force through the ROTC program in 1970. He entered active duty then and served as a Missile Launch Officer at Whiteman Air Force Base, Missouri, and as an Initial Qualification Training Instructor at Vandenberg Air Force Base, California, until being selected for the first Graduate Strategic and Tactical Sciences class in the School of Engineering, Air Force Institute of Technology in September 1977.

Permanent address: 6961 Leyden Street
Commerce City, Colorado 80022

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER AFIT/GST/MA/79M-1	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) SPECTRAL ANALYSIS OF TIME SERIES		5. TYPE OF REPORT & PERIOD COVERED MS Thesis 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Anthony L. Bertapelle Captain USAF		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Air Force Institute of Technology(AFIT/EN) Wright-Patterson AFB, Ohio 45433		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS		12. REPORT DATE March 1979 13. NUMBER OF PAGES 58
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

Approved for public release; IAW AFR 190-17

Timothy J. Miller, SSgt, USAF
NCOIC, Internal and Public Information

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Statistics
Spectral Density Matrix
Multivariate Analysis

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

One use of spectral analysis of time series is to determine if two different time series are realizations from the same process. This thesis develops the theory behind Krishnaiah and Schuurmann's theoretical work reported in their report Approximations to the Distributions of the Traces of Complex Multivariate Beta and F Matrices. We take the trace of a test statistic calculated from the spectral density matrices of the time

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

series and test it. The thesis applies the theory to two small sample simulations.